SDMC: Generating Functions- Solutions

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1. For some of the following exercises in this question, previous results will be used to do future problems.
(a) The generating function is: $1 + x^2 + x^4 + x^6 + ... = \frac{1}{1-x^2}$
(b) The expanded generating function is (Call it F(x) to make it easier to work with):

$$1 + 2x + 3x^2 + 4x^3 + ... = F(x)$$

$$x + 2x^2 + 3x^3 + ... = xF(x)$$

$$1 + x + x^2 + ... = (1 - x)(F(x))$$

$$\frac{1}{(1-x)^2} = F(x)$$

(c) The expanded generating function is:

$$1 + 3x + 6x^2 + 10x^3 + ... = F(x)$$

$$x + 3x^2 + 6x^3 + ... = x(F(x))$$

$$1 + 2x + 3x^2 + ... = (1 - x)F(x)$$

$$\frac{1}{(1-x)^3} = F(x)$$

(d) The expanded generating function is:

$$1 + 4x + 9x^2 + 16x^3 + ... = F(x)$$

$$x + 4x^2 + 9x^3 + 16x^4 + ... = xF(x)$$

$$1 + 3x + 5x^2 + 7x^3 + ... = (1 - x)(F(x))$$

$$x + 3x^2 + 5x^3 + ... = x(1 - x)(F(x))$$

$$1 + 2x(1 + x + x^2 + x^3 + ...) = (1 - x)^2(F(x))$$

$$F(x) = \frac{1+x}{(1-x)^3}$$

(e) Expanding the generating function again...

$$1 + 8x + 27x^2 + 64x^3 + ... = F(x)$$

$$x + 8x^2 + 27x^3 + ... = xF(x)$$
$$1 + 7x + 19x^2 + 37x^3 + ... = (1-x)(F(x))$$
$$x + 7x^2 + 19x^3 + ... = x(1-x)(F(x))$$
$$1 + 6x(1 + 2x + 3x^2 + 4x^3 + 5x^4 + ...) = (1-x)^2(F(x))$$
$$F(x) = \frac{1+4x+x^2}{(1-x)^4}$$

2. A problem like this requires careful consideration of the expanded formula and the recursion that is given... The generating function looks something like this:

$$a_0 + a_1x + a_2x^2 + ... = F(x)$$

$$2a_0x + 2a_1x^2 + 2a_2x^3 + ... = 2x(F(x))$$
$$a_0 + x + x^2 + x^3 + ... = (1-2x)(F(x))$$

When you subtract $2a_n$ from $a_{n+1} = 2a_n + 1$, you get 1 which was the motivation for multiplying the generating function by 2x instead of x. Now, we plug in our value for $a_0 = 0$ to get our final answer:

$$F(x) = \frac{x}{(1-2x)(1-x)}$$

3. Probably one of the most useful things you can do with generating functions: going back to the formula of the recursion. We use a technique called **partial fraction decomposition**. I will walk through it once in this problem and skip the technique in future problems and just show the answer.

$$F(x) = \frac{x}{(1-2x)(1-x)} = \frac{A}{1-x} + \frac{B}{1-2x}$$

Here, A and B are constants. The numerators of the partial fractions is some sort of a polynomial that has one less degree than the denominator. So for example if the denominator was a quadratic (See problem 5), then the numerator would be a linear polynomial of the form Ax+B. Now, we multiply both sides to clear the denominator:

$$x = (1-2x)A + B(1-x)$$

$$x = (A+B) - (2A+B)x$$

We know that the coefficients must be equal, so:

$$A + B = 0$$

$$-(2A+B) = 1$$

Solving we get that $A = -1$ and $B = 1$. So we know that the generating function is equal to:

$$\frac{1}{1-2x} - \frac{1}{1-x}$$

Expanding both of the generating functions, we get that:

$$F(x) = (1 + 2x + 4x^2 + 8x^3 + ...) - (1 + x + x^2 + x^3 + x^4 + ...)$$

So we know that the first term is the generating function for the sequence $b_n = 2^n$. The second term is the generating function for the sequence $c_n = 1$. So, the sequence we want to find the closed form of is the difference of those two sequences, so we have that $a_n = 2^n - 1$. For this particular problem, it probably would have been easier to compute the first couple of terms and then guess at the formula. Then we could have proved it using induction. That technique of guess and check works when the formula is relatively simple, but for more complicated sequences this technique becomes more desirable (see problems 5 and 12).

4. We'll use the same technique as we did in the previous problem:

$$F(x) = a_0 + a_1x + a_2x^2 + ...$$

$$(2x)(F(x)) = 2a_0x + 2a_1x^2 + 2a_2x^3 + ...$$

$$(1-2x)(F(x)) = a_0 + x^2 + 2x^3 + 3x^4 + 4x^5 + ...$$

$$(1-2x)(F(x)) = 1 + x^2(1 + 2x + 3x^2 + ...)$$

$$(1-2x)(F(x)) = 1 + \frac{x^2}{(1-x)^2}$$

$$F(x) = \frac{2x^2 - 2x + 1}{(1-x)^2(1-2x)}$$

5. Using the partial fraction decomposition technique again, we see that:

$$F(x) = \frac{2}{1-2x} - \frac{1}{(1-x)^2}$$

You might be wondering how the denominators are chosen. Sometimes, you need to take multiple tries to get a decomposition that works and you can make sense out of it. As with a lot of other things in math, choosing the denominators in the partial fraction decomposition takes practice. From the decomposition:

$$F(x) = 2(1 + 2x + 4x^2 + 8x^3 + ...) - (1 + 2x + 3x^2 + 4x^3 + ...)$$

Examining the sequences that the generating functions correspond to individually, we can see that the formulat that we want is:

$$a_n = 2^{n+1} - (n+1) = 2^{n+1} - n - 1$$

6. We approach this problem just like we approached the problem that we did in the notes. The answer is:

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})}$$

7. a) This can be done by calculating the x^{10} in the generating function. In my opinion, this is one of those problems where generating functions doesn't serve you that well since it is the same as just brute forcing the answer or using another approach. Anyways, you should get that the answer is 6.
b) This is done just like Problem 6. The answer is:

$$\frac{1}{(1-x)(1-x^2)}$$

8. In this problem, generating functions is one of the best way to do this problem. So, let's look at each part and find the generating functions for each. Then we have proven that they are equal when we see that the generating functions are equal.

So the first part is: the generating function for the number of partitions of n into parts so that the largest of them is r. We know the generating function for the number of partitions of n, and we want the number of partitions up to r. So that is just the generating function:

$$\prod_{n=1}^{r}(\frac{1}{1-x^n})$$

That is not quite right, because if you look at part that determines the number of r's chosen, we see that the generating function is:

$$1 + x^r + x^{2r} + x^{3r} + ... = \frac{1}{1-x^r}$$

If we take the 1 from this generating function, then we don't have r in the corresponding partition! So, we're not allowed to take 1 and we must not include that option. The generating function thus becomes:

$$x^r + x^{2r} + x^{3r} + ... = \frac{x^r}{1-x^r}$$

So, the generating function for the first part is:

$$x^r \prod_{n=1}^{r}(\frac{1}{1-x^n})$$

Now we just have to find the generating function for the second part and we are done. So the number of partitions of any number n into exactly r parts. Think about forming the partitition this way. First, we make all of the r parts equal to 1, since they must all at least be 1. Then, we make partitions normally and match terms. For concreteness, lets look at an example. Say r = 3. First, we'll start with the partition 1+1+1. So if n = 5, we'll look at a partition of n-r = 2. 2 = 1+1. We'll add the first element of the original partition with the first element of the partition of 2. Then the second elemtents will be added, and so on. So we get that 2+2+1 is a partition of 5 into 3 parts. Then, we'll count that. Giving 1 to each term is like multiplying the generating function by x^r. So we want the partitions of n-r into r parts shifted by x^r. So the generating function is:

$$x^r \prod_{n=1}^{r}(\frac{1}{1-x^n})$$

Since the generating function for the first part is equal to the generating function for the second part, the two quantities are equal for all n.

9. a) Since all of the parts are distinct, we can't take two of a part. Therefore every generating function stops at $1+x^n$.

$$\prod_{n=1}^{\infty}(1+x^n)$$

b)

$$\prod_{n=1}^{\infty}(\frac{1}{1-x^{2n-1}})$$

This is the same as finding the generating function for all partitions except we're only allowed to use the odd parts now.
c)

$$\prod_{n=1}^{k}(\frac{1}{1-x^n})$$

This follows the same argument as the layout argument in the second part of the previous problem. This time, we are laying out the first partition 0+0+0+0+0...+0 (k times) and then counting the number of partitions that we can place on top of that.
d)

$$x^k\prod_{n=1}^{k}(\frac{1}{1-x^k})$$

e) Look at the number of ways of representing n as 2x with x being an integer, and so on for the other two terms. Applying all of the restrictions correctly, you should get:

$$\frac{1+x^7+x^{14}+x^{21}}{(1-x^2)(1-x^3)}$$

10. The number of partitions where each part is different is:

$$\prod_{n=1}^{\infty}(1+x^n)$$

This is the generating function because I can't use more than 1 of each number since it is into distinct partitions.

$$\prod_{n=1}^{\infty}\left(\frac{1}{1-x^{2n-1}}\right)$$

We want the regular generating function, but we don't want the odd terms.

So, to manipulate the generating functions to look like each other:

$$\prod_{n=1}^{\infty}(1+x^n)$$

$$= (1+x)(1+x^2)(1+x^3)...$$

$$= \frac{(1-x^2)}{1-x}\frac{(1-x^4)}{1-x^2}\frac{(1-x^6)}{1-x^3}+...$$

$$= \prod_{n=1}^{\infty}\left(\frac{1}{1-x^{2n-1}}\right)$$

We multiplied the original generating function by the conjugate to get a denominator. Then we cancelled to get our answer.

11. So, we look at the generating function for the number of partitions of a number.

$$p(n) = \prod_{n=1}^{\infty}\left(\frac{1}{1-x^n}\right)$$

To count the number of 1's in the partition, we must replace the generating function for the number of 1's in the partition $(1+x+x^2+...)$ with something else. If we think about what we are doing, then it becomes clear what we need to replace it with. If we use the x^0 term, we want to count that 0 times. If we use the x^1 we want to count that 1 time. If we use the x^2 term, we want to count that 2 times. And so on. Therefore, that generating function changes to:

$$0+x+2x^2+3x^3+... = \frac{1}{(1-x)^2} = \frac{x}{1-x}\frac{1}{1-x}$$

So to fix the original function for the number of partitions of a number to count the number of 1's instead, we must multiply it by $\frac{x}{1-x}$. So the generating function for that is:

$$\frac{x}{1-x}\prod_{n=1}^{\infty}\frac{1}{1-x^n}$$

Instead of counting the number of distinct terms of a partition, we will count the number of partitions that 1 appears in. Then, we'll count the number of partitions of n that 2 appears in. And so on. So, we'll do complementary counting to do that. We'll count the number of times that 1 does not appear and subtract it from the total number of partitions of n to get the number of partitions of n that 1 does appear in. Then, we'll do this process for all n and add it up to get our answer. So, we want to evaluate this:

$$\sum_{n=1}^{\infty}\left(p(n)-\frac{p(n)}{\frac{1}{1-x^n}}\right)$$

$$=\sum_{n=1}^{\infty}(p(n)x^n)$$

$$=p(n)\sum_{n=1}^{\infty}(x^n)$$

$$=\frac{x}{1-x}p(n)$$

$$=\frac{x}{1-x}\prod_{n=1}^{\infty}\left(\frac{1}{1-x^n}\right)$$

So, we have proven that those two generating functions are equal, therefore we are done.

12. We can look at the number of polynomials by considering an analogous question. How many ways are there of expressing n as a sum of powers of 2, using each power of 2 a maximum of 3 times only? That is an much easier generating function question:

$$(1+x+x^2+x^3)(1+x^2+x^4+x^6)(1+x^4+x^8+x^12)...$$

$$=\frac{1-x^4}{1-x}\frac{1-x^8}{1-x^2}\frac{1-x^{16}}{1-x^4}...$$

$$=\frac{1}{(1-x)(1-x^2)}$$

$$=\frac{-\frac{1}{4}x+\frac{3}{4}}{(1-x)^2}+\frac{\frac{1}{4}}{1+x}$$

$$=\frac{-x}{4}(1+2x+3x^2+...)+\frac{3}{4}(1+2x+3x^2+...)+\frac{1}{4}(1-x+x^2-x^3+x^4...)$$

Therefore $a_n=\frac{n}{2}+\frac{3}{4}+\frac{(-1)^n}{4}$

13. Instead of 19, consider the sequence of the number of solutions for all n. This is somewhat a partition problem. Now let us find the generating function. I will organize my work as such. I will put the term that we are concerned with on the left hand side of the equation, and the generating function for the term on the right hand side.

$$a_1 = 1 + x = \frac{1-x^2}{1-x}$$

$$a_2 = 1 + x + x^2 = \frac{1-x^3}{1-x}$$

$$a_3 = 1 + x + x^2 + x^3 = \frac{1-x^4}{1-x}$$

$$a_4 = 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x}$$

$$2b_1 = \frac{1}{1-x^2}$$

$$3b_2 = \frac{1}{1-x^3}$$

$$4b_3 = \frac{1}{1-x^4}$$

$$5b_4 = \frac{1}{1-x^5}$$

Multiplying all of the individual generating functions together, we get that the generating function is: $\frac{1}{(1-x)^4}$. You can choose to find the x^{19} coefficient in many ways. You should get 1540 an your answer.

14. Since the partitions are ordered, more complexities come into play. Think about splitting the partitions in terms of the number of elements that it has. If it has 0 elements in it (0 1's and 2's), then there is only 1 way to do it. If it has 1 element in it (1 1 or 1 2), then there is only 1 way to make a 1 and one way to make a 2, so the generating functon for this part is $(x + x^2)$. It the partition has 2 elements in it, the generating function is $(x + x^2)^2$ since we care about the order that we choose the 1 and the 2 in. So, you can now begin to see the pattern. We add the generating functions together since we were considering the partitions as a whole. So the generating function is:

$$1 + (x + x^2) + (x + x^2)^2 + ... = \frac{1}{1-x-x^2}$$

Now we must do the same thing for the second part. So this generating function is the same as the previous one, but instead of having $x+x^2$ repeat over and over, we have $(x^3+x^4+x^5+...)$ that repeats over and over. Also, using no numbers is out of the question since we are looking at n+2 which must always have a partition. This is because all of the partitions that we are counting are ordered. So this generating function is:

$$(x^3+x^4+x^5+...)+(x^3+x^4+x^5+...)^2+...$$

$$=\frac{\frac{x^2}{1-x}}{1-\frac{x^2}{1-x}}$$

$$=\frac{x^2}{1-x-x^2}$$

This is the sequence shifted over two since we are looking at n+2. So we must divide by x^2 to get our desired generating function for the second part. Since the generating functions are equal, the two quantities that we want to compare are equal. As a side note, this generating function is the generating function for the Fibonacci sequence.

15. The generating function for number of partitions that aren't multiples of 3 are:

$$\frac{\prod_{n=1}^{\infty}\frac{1}{1-x^n}}{\prod_{n=1}^{\infty}\frac{1}{1-x^{3n}}}$$

The genertaint function for the number of partitions of n where there are at most 2 parts is:

$$(1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)...$$

$$=\frac{1-x^3}{1-x}\frac{1-x^6}{1-x^2}\frac{1-x^9}{1-x^3}$$

$$=\frac{\prod_{n=1}^{\infty}\frac{1}{1-x^n}}{\prod_{n=1}^{\infty}\frac{1}{1-x^{3n}}}$$